Crossover Behavior in Burst Avalanches of Fiber Bundles: Signature of Imminent Failure

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Bundles of many fibers, with statistically distributed thresholds for breakdown of individual fibers and where the load carried by a bursting fiber is equally distributed among the surviving members, are considered. During the breakdown process, avalanches consisting of simultaneous rupture of several fibers occur, with a distribution $D(\Delta)$ of the magnitude Δ of such avalanches. We show that there is, for certain threshold distributions, a crossover behavior of $D(\Delta)$ between two power laws $D(\Delta) \propto \Delta^{-\xi}$, with $\xi = 3/2$ or $\xi = 5/2$. The latter is known to be the generic behavior, and we give the condition for which the $D(\Delta) \propto \Delta^{-3/2}$ behavior is seen. This crossover is a signal of imminent catastrophic failure in the fiber bundle. We find the same crossover behavior in the fuse model.

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A fundamental question in strength considerations of materials is when does it fail. Are there signals that can warn of imminent failure? This is of uttermost importance in e.g. the diamond mining industry where sudden failure of the mine can be extremely costly in terms of lives. These mines are under continuous acoustic surveillance, but at present there are no tell-tale acoustic signature of imminent catastrophic failure. The same type of question is of course also central to earth quake prediction. We will in this letter study signatures of imminent failure in the context of the *fiber bundle model* [1]. We find that if a histogram of the number of fibers failing simultaneously is recorded over an interval which starts sometime during the failure process, it follows a power law with an exponent that crosses over to a very different value if the start of the interval is close enough to the point at which the fiber bundle fails catastrophically. This is a clear signature of imminent failure. We also study the fuse model in this context and find that it behaves qualitatively in the same manner as the fiber bundle model.

When a weak element in a loaded material fails, the increased stress on the remaining elements may cause further failures, and thereby give a burst avalanche in which Δ elements fail simultaneously. With further increase in the load new avalanches occur. A bundle of many fibers with stochastically distributed fiber strengths, and clamped at both ends, is a much studied model for such avalanches. In its classical version [1], a ruptured fiber carries no load and the increased stresses caused by a failed element are shared equally by all surviving fibers.

A main result [2, 3] for this model is that under mild restrictions on the fiber strength distribution the expected number $D(\Delta)$ of burst avalanches of size Δ is governed

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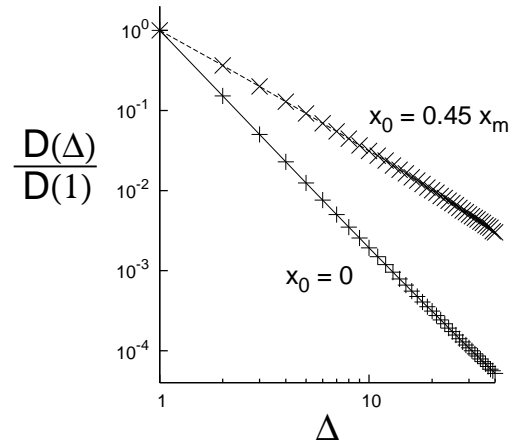‡Electronic address: Per.Hemmer@phys.ntnu.no

FIG. 1: The distribution of bursts for the strength distribution (2) with $x_0 = 0$ and $x_0 = 0.45x_m$. The figure is based on 10000 samples with $N = 50000$ fibers.

by a universal power law

$$D(\Delta) \propto \Delta^{-\xi} \tag{1}$$

for large Δ, with $\xi = 5/2$.

When the load on the bundle is increased beyond a critical threshold x_c, the whole bundle ruptures [4, 5]. We are interested in the final stages of the breakdown process, when the N surviving fibers has a distribution of fiber strengths in the interval (x_0, x_m), with x_0 slightly less than x_c. Assume first that the fiber strengths in which the maximal loads x_n that the fibers $n = 1, 2, \ldots N$ are able to carry are picked independently with a probability density

$$p(x) = \text{Prob}(x \leq x_n < x + dx)/dx$$

$$= (x_m - x_0)^{-1} \quad \text{for } x_0 \leq x \leq x_m \tag{2}$$

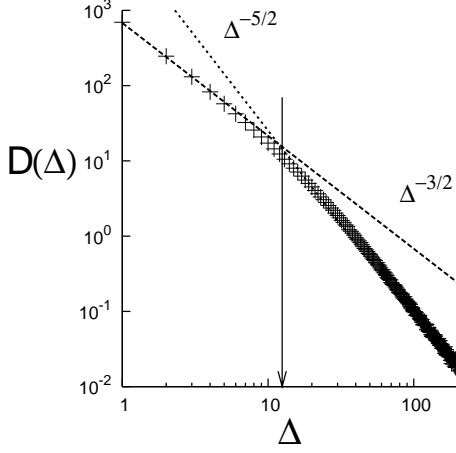When x_0 is greater than the critical value $x_c = x_m/2$ the

FIG. 2: The distribution of bursts for the strength distribution (Eqn. 2) with $x_0 = 0.40x_m$. The figure is based on 10000 samples with $N = 50000$ fibers. The straight lines represent two different power laws and the arrow locates the crossover point $\Delta_c \simeq 12.5$.

whole bundle fails fatally when the load per fiber exceeds x_0. This can be seen as follows: The force $F(x)$ that the bundle is able to withstand when all fibers with strengths less than x have ruptured, is proportional to the number of surviving fibers times the strength,

$$
F(x) \propto x \int_x^{x_m} p(x) \, dx
$$
$$
= \frac{x(x_m - x)}{x_m - x_0} \quad \text{for } x_0 < x \leq x_m . \tag{3}
$$

When $x_0 > x_c = x_m/2$, this is a decreasing function of x. The consequence in this case is that after the first fiber ruptures at $x = x_0$ the whole bundle will fail completely at once [5]. A threshold distribution in which the weakest fiber has the critical value we call a critical threshold distribution. We want to study the burst distribution close to this critical situation. In Fig. 1 we show results for $D(\Delta)$ in a simulation experiment on fiber bundles with the strength distribution (2), with $x_0 = 0.45x_m = 0.9x_c$. For comparison, results with $x_0 = 0$ are shown. In both cases $D(\Delta)$ shows a power law decay, apparently with an exponent $\xi = 5/2$ for the $x_0 = 0$ case, and an exponent $3/2$ for $x_0 = 0.45x_m$ (bundle with no weak fibers) [5]. In this note we explain the latter result as a crossover phenomenon.

For a bundle of many fibers the expected number of bursts of magnitude Δ is given by [2]

$$
\frac{D(\Delta)}{N} =
$$
$$
\frac{\Delta^{\Delta-1}}{\Delta!} \int_0^{x_c} p(x) \left[1 - xp(x)/Q(x)\right] \left[xp(x)/Q(x)\right]^{\Delta-1}
$$
$$
\times \exp\left[-\Delta x p(x)/Q(x)\right] dx , \tag{4}
$$

where $Q(x) = \int_x^\infty p(y) \, dy$ is the fraction of fibers with strength exceeding x. For the distribution (2) this yields

$$
\frac{D(\Delta)}{N} = \frac{\Delta^{\Delta-1}}{\Delta!(x_m - x_0)}
$$
$$
\times \int_{x_0}^{x_c} \frac{x_m - 2x}{x} \left[\frac{x}{x_m - x} e^{-x/(x_m - x)}\right]^\Delta \, dx . \tag{5}
$$

Introducing the parameter

$$
\epsilon = \frac{x_c - x_0}{x_m} \tag{6}
$$

and a new integration variable

$$
z = \frac{x_m - 2x}{\epsilon(x_m - x)} , \tag{7}
$$

we obtain

$$
\frac{D(\Delta)}{N} = \frac{2\Delta^{\Delta-1}e^{-\Delta}\,\epsilon^2}{\Delta!(1 + 2\epsilon)}
$$
$$
\times \int_0^{4/(1+2\epsilon)} \frac{z}{(1-\epsilon z)(2-\epsilon z)^2} \, e^{\Delta[\epsilon z + \ln(1-\epsilon z)]} \, dz. \tag{8}
$$

For small ϵ, i.e., close to the critical threshold distribution, we expand

$$
\epsilon z + \ln(1 - \epsilon z) = -\tfrac{1}{2}\epsilon^2 z^2 - \tfrac{1}{3}\epsilon^3 z^3 + \dots , \tag{9}
$$

with the result

$$
\frac{D(\Delta)}{N} \simeq \frac{\Delta^{\Delta-1}e^{-\Delta}\epsilon^2}{2\Delta!}
$$
$$
\times \int_0^4 e^{-\Delta\epsilon^2 z^2/2} \, z dz = \frac{\Delta^{\Delta-2}e^{-\Delta}}{2\Delta!}\left(1 - e^{-8\epsilon^2\Delta}\right) . \tag{10}
$$

By use of Stirling approximation

$$
\Delta! \simeq \Delta^\Delta e^{-\Delta}\sqrt{2\pi\Delta} , \tag{11}
$$

— a reasonable approximation even for small Δ — this may be written

$$
\frac{D(\Delta)}{N} \simeq (8\pi)^{-1/2}\Delta^{-5/2}\left(1 - e^{-\Delta/\Delta_c}\right) , \tag{12}
$$

with

$$
\Delta_c = \frac{1}{8\epsilon^2} = \frac{x_m^2}{8(x_c - x_0)^2} . \tag{13}
$$

We see from (12) that there is a crossover at a burst length around Δ_c, so that

$$
\frac{D(\Delta)}{N} \simeq \begin{cases} (8/\pi)^{1/2}\epsilon^2 \, \Delta^{-3/2} & \text{for } \Delta \ll \Delta_c \\ (8\pi)^{-1/2}\Delta^{-5/2} & \text{for } \Delta \gg \Delta_c . \end{cases} \tag{14}
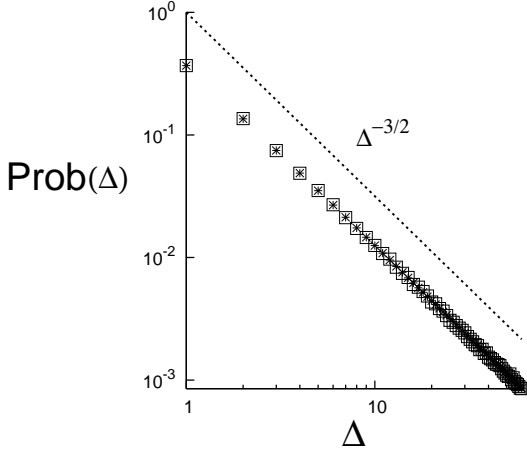$$

FIG. 3: Distribution of first burst for the critical strength distribution (Eqn. 2) with $x_0 = 0.50x_m$. The simulation results (squares) are based on 10^6 samples with $N = 50000$ fibers. The 'star' symbol stands for the analytic result (Eqn. 18).

For $x_0 = 0.45x_m$, we have $\Delta_c = 50$, so the final asymptotic behavior is not visible in Fig. 1. The crossover is seen better for $x_0 = 0.40x_m$. In Fig. 2 there is clearly a crossover near $\Delta = \Delta_c = 12.5$.

The phenomenon is not limited to the uniform threshold distribution. The $\xi = 3/2$ power law [5] in the burst size distribution will appear whenever a threshold distribution is non-critical, but close to criticality. This can be seen from the expression for the average force $F(x) \propto xQ(x)$ on the bundle. The critical value x_c corresponds to the maximum of $F(x)$, which gives $Q(x_c) = x_c p(x_c)$. For the weakest fiber strength x_0 very close to x_c and Δ finite, the integrand in (4) therefore approach a constant times $e^{-\Delta}(x_c - x)$. Then $D(\Delta)$ is proportional to $\Delta^{\Delta-1}e^{-\Delta}/\Delta! \simeq (2\pi)^{-1/2}\Delta^{-3/2}$. On the other hand, when $\Delta \gg (x_c - x_0)^{-2}$ the generic asymptotic behavior $D \propto \Delta^{-5/2}$ will follow.

Precisely at criticality $\Delta_c = \infty$ and the $\xi = 5/2$ power law is no longer present. We can demonstrate by a random walk argument that *at criticality* the burst distribution follows a $3/2$ power law. The load on the bundle when the kth fiber with strength x_k is about to fail is

$$F_k = Q(x_k)x_k, \qquad (15)$$

and the fluctuations of this load determine the size of the bursts. The probability $\rho(f)\,df$ that the difference $F_{k+1} - F_k$ is in the interval $(f, f + df)$ has been shown to be [3]

$$\rho(f) = \begin{cases} \frac{p(x_k)}{Q(x_k)}\,e^{-(f+x_k)p(x_k)/Q(x_k)} & \text{for } f \geq -x_k \\ 0 & \text{for } f < -x_{ka} . \end{cases} \qquad (16)$$

At criticality $x\,p(x) = Q(x)$, which gives

$$\rho(f) = \begin{cases} x^{-1}\,e^{-1-f/x} & \text{for } f \geq -x \\ 0 \text{ for } f < -x . \end{cases} \qquad (17)$$
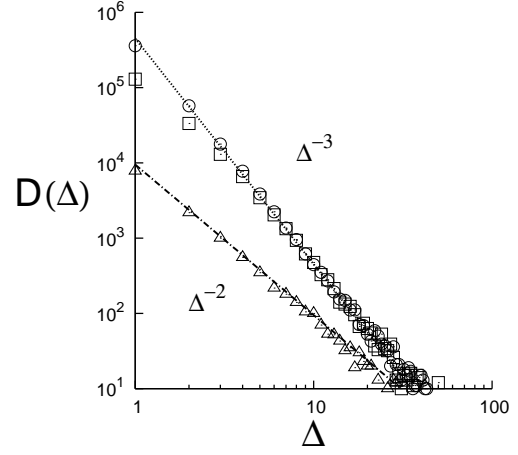

FIG. 4: The burst distribution in the fuse model. System size is 100×100 and the number of samples was 300. On the average, catastrophic failure sets in after 2097 fuses have blown. The circles denote the burst distribution measured throughout the entire breakdown process. The squares denote the burst histrogram based on those appearing after the first 1000 fuses have blown. The triangles denote the burst histogram after 2090 fuses have blown. The two straight lines indicate power laws with exponents -3.0 and -2.0 respectively.

This can be considered as the step probability in a random walk. The random walk is unsymmetrical, but it is *unbiased*, $< f > = 0$, as it should be at criticality.

Using the step probability (17) this may be evaluated [6], with the result

$$\text{Prob}(\Delta) = \frac{\Delta^{\Delta-1}}{e^{\Delta}\,\Delta!}. \qquad (18)$$

The simulation results in Fig. 3 are in excellent agreement with this distribution. At the completion of a burst the force, i.e., the excursion of the random walk, is larger than all previous values. Therefore one may use this point as a new starting point to find, by the same calculation, the distribution of the next burst, etc. Consequently the complete burst distribution at criticality is proportional to (18), i.e. essentially $\propto \Delta^{-3/2}$.

In order to test this crossover phenomenon in a more complex situation, we have studied the burst distribution in the fuse model [7]. The fuse model consists of a lattice where each bond is an ohmic resistor as long as the electrical current it carries is below a threshold value. If the threshold is passed, the fuse burns out irreversibly. The threshold t of each bond is drawn from an uncorrelated distribution $p(t)$. The lattice is placed between electrical bus bars and an increasing current is passed through it. Numerically, the Kirchhoff equations are solved with a voltage difference between the bus bars set to unity. The ratio between current i_j and threshold t_j for each bond j is calculated and the bond having the largest value, $\max_j(i_j/t_j)$ is identified and subsequently irreversibly removed. The lattice is a two-dimensional

square one placed at 45° with regards to the bus bars. The threshold distribution is uniform on the unit interval. All fuses have the same resistance. The burst distribution follows the power law (1) with $\xi = 3.0$, which is consistent with the value determined by Hansen and Hemmer [8]. We show the histogram in Fig. 4. With a system size of 100×100, 2097 fuses blow on the average before catastrophic failure sets in. When measuring the burst distribution only after the first 2090 fuses have blown, a different power law is found, this time with $\xi = 2.0$. After 1000 blown fuses, on the other hand, ξ remains the same as for the histogram recording the entire failure process, see Fig. 4.

In conclusion, we have studied the burst distribution in the fiber bundle model and shown that close to catastrophic failure it exibits a crossover behavior with two power law with exponents -3/2 and -5/2. In the critical situation a random walk argument gives a single power law with exponents -3/2. We show numerically, that the same crossover — but with different values for the exponents — in the two-dimensional fuse model. This crossover signals that catastrophic failure is imminent. This signal does not hinge on observing rare events, and therefore has a strong potential as a useful detection tool.

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